Generalisation of the Danilov-Karzanov-Koshevoy Construction for Peak-Pit Condorcet Domains

ARKADII SLINKO

Danilov, Karzanov and Koshevoy (2012) geometrically introduced an interesting operation of composition on Condorcet domains and using it they disproved a long-standing problem of Fishburn about the maximal size of connected Condorcet domains. We give an algebraic definition of this operation and investigate its properties. We give a precise formula for the cardinality of composition of two Condorcet domains and improve the Danilov, Karzanov and Koshevoy result showing that Fishburn's alternating scheme does not always produce a largest connected Condorcet domain.

1 INTRODUCTION

The famous Condorcet Paradox shows that if voters' preferences are unrestricted, the majority voting can lead to intransitive collective preference in which case the Condorcet Majority Rule [Condorcet, 1785], despite all its numerous advantages, is unable to determine the best alternative, i.e., it is not always decisive. Domain restrictions is, therefore, an important topic in economics and computer science alike [Elkind, 2018]. In particular, for artificial societies of autonomous software agents there is no problem of individual freedom and, hence, for the sake of having transitive collective decisions the designers can restrict choices of those artificial agents in order to make the majority rule work.

Condorcet domains are sets of linear orders with the property that, whenever the preferences of all voters belong to this set, the majority relation of any profile with an odd number of voters is transitive. Maximal Condorcet domains historically have attracted a special attention since they represent a compromise which allows a society to always have transitive collective preferences and, under this constraint, provide voters with as much individual freedom as possible. The question: "How large a Condorcet domain can be?" has attracted even more attention (see the survey of Monjardet [2009] for a fascinating account of historical developments). Kim et al. [1992] identified this problem as a major unsolved problem in the mathematical social sciences. Fishburn [1996] introduced the function

$$f(n) = \max\{|\mathcal{D}| : \mathcal{D} \text{ is a Condorcet domain on the set of } n \text{ alternatives.}\}$$

and put this problem in the mathematical perspective.

Abello [1991] and Fishburn [1996, 2002] managed to construct some "large" Condorcet domains based on different ideas. Fishburn, in particular, taking a clue from Monjardet example (sent to him in private communication), came up with the so-called alternating scheme domains (that will be defined later in the text), later called Fishburn's domains [Danilov et al., 2012]. This scheme produced Condorcet domains with some nice properties, which, in particular, are connected and have maximal width (see the definitions of these concepts later in this paper). Fishburn [1996] conjectured (Conjecture 2) that among Condorcet domains that do not satisfy the so-called never-middle condition (these in [Danilov et al., 2012] were later called peak-pit domains), the alternating scheme provides domains of maximum cardinality. Galambos and Reiner [2008] formulated another similar hypothesis (Conjecture 1) which later appeared to be equivalent to Fishburn's one [Danilov et al., 2012]. Monjardet [2006] introduced the function

$$g(n) = \max\{|\mathcal{D}| : \mathcal{D} \text{ is a peak-pit Condorcet domain on the set of } n \text{ alternatives}\}$$

in terms of which Fishburn's hypothesis becomes $g(n) = |F_n|$, where F_n is the nth Fishburn domain. Monjardet [2009] also emphasised Fishburn's hypothesis.

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It is known that $g(n) = f(n)$ for $n \leq 7$ [Fishburn, 1996, Galambos and Reiner, 2008] and it is believed that $g(16) < f(16)$ [Monjardet, 2009]. This is because [Fishburn, 1996] showed that $f(16) > |F_{16}|$. Thus, if Fishburn's hypothesis were true we would get $f(n) > g(n)$ for large n. However, this hypothesis is not true.

Danilov et al. [2012] introduced the class of tiling domains which are peak-pit domains of maximal width and defined an operation on tiling domains that allowed them to show that $g(42) > |F_{42}|$. This operation was somewhat informally defined which made investigation of it and application of it in other situations difficult. In the present article we give an algebraic definition and a generalisation of the Danilov-Karzanov-Koshevoy construction and investigate its properties. In our interpretation it involves two peak-pit Condorcet domains $\mathcal{D}_1$ and $\mathcal{D}_2$ on sets of n and m alternatives, respectively, and two linear orders $u \in \mathcal{D}_1$ and $v \in \mathcal{D}_2$; the result is denoted as $(\mathcal{D}_1 \otimes \mathcal{D}_2)(u, v)$. It is again a peak-pit Condorcet domain on $n+m$ alternatives which exact cardinality we can calculate. Using this formula we can slightly refine the argument from [Danilov et al., 2012] to show that $g(40) > |F_{40}|$.

2 PRELIMINARIES

Let A be a finite set and $\mathcal{L}(A)$ be the set of all (strict) linear orders on A. Any subset $\mathcal{D} \subseteq \mathcal{L}(A)$ will be called a *domain*. Any sequence $P = (v_1, \ldots, v_n)$ of linear orders from $\mathcal{D}$ will be called a *profile* over $\mathcal{D}$. A linear order $a_1 > a_2 > \cdots > a_n$ on A, will be denoted by a string $a_1a_2 \ldots a_n$. Let us also introduce notation for reversing orders: if $x = a_1a_2 \ldots a_n$, then $\bar{x} = a_na_{n-1} \ldots a_1$. If linear order v_i ranks a higher than b, we denote this as $a \succ_i b$.

Definition 1. *The* majority relation $\succeq_P$ *of a profile P is defined as*

$$a \succeq_P b \iff |\{i \mid a \succ_i b\}| \geq |\{i \mid b \succ_i a\}|.$$

For an odd number of linear orders in the profile P this relation is a tournament, i.e., complete and asymmetric binary relation. In this case we denote it $\succ_P$.

Now we can define the main object of this investigation.

Definition 2. *A domain $\mathcal{D} \subseteq \mathcal{L}(A)$ over a set of alternatives A is a* Condorcet domain *if the majority relation of any profile P over $\mathcal{D}$ with odd number of voters is transitive. A Condorcet domain $\mathcal{D}$ is* maximal *if for any Condorcet domain $\mathcal{D}' \subseteq \mathcal{L}(A)$ the inclusion $\mathcal{D} \subseteq \mathcal{D}'$ implies $\mathcal{D} = \mathcal{D}'$.*

There is a number of alternative definitions of Condorcet domains, see e.g., Monjardet [2009], Puppe and Slinko [2019].

Up to an isomorphism, there is only one maximal Condorcet domain on the set $\{a, b\}$, namely $CD_2 = \{ab, ba\}$ and there are only three maximal Condorcet domains on the set of alternatives $\{a, b, c\}$, namely,

$$CD_{3,t} = \{abc, acb, cab, cba\},\ CD_{3,m} = \{abc, bca, acb, cba\},\ CD_{3,b} = \{abc, bac, bca, cba\}.$$

The first domain contains all the linear orders on a, b, c where c is never ranked first, second contains all the linear orders on a, b, c where a is never ranked second and the third contains all the linear orders on a, b, c where b is never ranked last. Following Monjardet, we denote these conditions as $cN_{\{a,b,c\}}1$, $aN_{\{a,b,c\}}2$ and $bN_{\{a,b,c\}}3$, respectively. We note that these are the only conditions of type $xN_{\{a,b,c\}}i$ with $x \in \{a, b, c\}$ and $i \in \{1, 2, 3\}$ that these domains satisfy.

A domain that for any triple $a, b, c \in A$ satisfies a condition $xN_{\{a,b,c\}}1$ with $x \in \{a, b, c\}$ is called *never-top* domain, a domain that for any triple $a, b, c \in A$ satisfies a condition $xN_{\{a,b,c\}}2$ with $x \in \{a, b, c\}$ is called *never-middle* domain, and a domain that for any triple $a, b, c \in A$ satisfies a condition $xN_{\{a,b,c\}}3$ with $x \in \{a, b, c\}$ is called *never-bottom* domain. A domain that for any triple satisfies either never-top or never-bottom condition is called *peak-pit domain* [Danilov et al., 2012]. Both never-top and never-bottom conditions will be called *peak-pit conditions.*

We note that Danilov et al. [2012], who consider linear orders over $A = \{1, 2, \ldots, n\}$, restrict the class of peak-pit domains to domains that contain two completely reversed orders (up to an isomorphism they can be taken as $12 \ldots n$ and $\overline{12 \ldots n} = n\, n-1 \ldots 1$) and prove that under this restriction all of them can be embedded into tiling domains (Theorem 2 of [Danilov et al., 2012]). We also note that never-bottom domains are also known as Arrow's single-peaked domains and maximal domains among them have all cardinality 2^{n-1} [Slinko, 2019].

Given a set of alternatives A, we say that

$$\mathcal{N} = \{xN_{\{a,b,c\}}i \mid \{a, b, c\} \subseteq A,\ x \in \{a, b, c\} \text{ and } i \in \{1, 2, 3\}\} \tag{1}$$

is a *complete set of never conditions* if it contains at least one never condition for every triple a, b, c of distinct elements of A. If the set of linear orders that satisfy $\mathcal{N}$ is non-empty, we say that $\mathcal{N}$ is consistent.

Proposition 1. *A domain of linear orders $\mathcal{D} \subseteq \mathcal{L}(A)$ is a Condorcet domain if and only if it is non-empty and satisfies a complete set of never conditions.*

Proof. This is well-known characterisation noticed by many researchers. See, for example, Theorem 1(d) in [Puppe and Slinko, 2019] and references there. □

This proposition, in particular, means that the collection $\mathcal{D}(\mathcal{N})$ of all linear orders that satisfy a certain complete set of never conditions $\mathcal{N}$, if non-empty, is a Condorcet domain. Let us also denote by $\mathcal{N}(\mathcal{D})$ the set of all never conditions that are satisfied by all linear orders from a domain $\mathcal{D}$.

Let $\psi\colon A \to A'$ be a bijection between two sets of alternatives. It can then be extended to a mapping $\psi\colon \mathcal{L}(A) \to \mathcal{L}(A')$ in two ways: by mapping linear order $u = a_1a_2 \ldots a_m$ onto $\psi(u) = \psi(a_1)\psi(a_2) \ldots \psi(a_m)$[1] or to $\overline{\psi(u)} = \psi(a_m)\psi(a_{m-1}) \ldots \psi(a_1)$.

Definition 3. *Let A and A' be two sets of alternatives (not necessarily distinct) of equal cardinality. We say that two domains, $\mathcal{D} \subseteq \mathcal{L}(A)$ and $\mathcal{D}' \subseteq \mathcal{L}(A')$ are* isomorphic *if there is a bijection $\psi\colon A \to A'$ such that $\mathcal{D}' = \{\psi(d) \mid d \in \mathcal{D}\}$ and flip-isomorphic if $\mathcal{D}' = \{\overline{\psi(d)} \mid d \in \mathcal{D}\}$.*

Example 1. *The single-peaked and single-dipped maximal Condorcet domains on $\{a, b, c\}$ are $CD_{3,b} = \{abc, bac, bca, cba\}$ and $CD_{3,t} = \{abc, acb, bca, bac\}$, respectively. They are not isomorphic but flip-isomorphic under the identity mapping of $\{a, b, c\}$ onto itself.*

Definition 4 ([Puppe, 2018]). *A Condorcet domain $\mathcal{D}$ is said to have* maximal width *if it contains two completely reversed orders, i.e., together with some linear order u it also contains $\bar{u}$.*

Up to an isomorphism, for any Condorcet domain $\mathcal{D}$ of maximal width we may assume that $A = \{1, 2, \ldots, n\}$ and it contains linear orders $e = 12 \ldots n$ and $\bar{e} = n \ldots 21$.

The universal domain $\mathcal{L}(A)$ is naturally endowed with the following betweenness structure (as defined by Kemeny [1959]). An order v is *between* orders u and w if $v \supseteq u \cap w$, i.e., v agrees with all binary comparisons in which u and w agree (see also [Kemeny and Snell, 1960]).The set of all orders that are between u and w is called the *interval* spanned by u and w and is denoted by $[u, w]$. The domain $\mathcal{L}(A)$ endowed with this betweenness relation is referred to as the *permutahedron* [Monjardet, 2009].

Given a domain of preferences $\mathcal{D}$, for any $u, w \in \mathcal{D}$ we define the induced interval as $[u, w]_{\mathcal{D}} = [u, w] \cap \mathcal{D}$. Puppe and Slinko [2019] defined a graph $G_{\mathcal{D}}$ associated with this domain. The set of linear orders from $\mathcal{D}$ are the set of vertices $V_{\mathcal{D}}$ of $G_{\mathcal{D}}$, and for two orders $u, w \in \mathcal{D}$ we draw an edge between them if there is no other vertex between them, i.e., $[u, w]_{\mathcal{D}} = \{u, w\}$. The set of edges

[1] We use the same notation for both mappings since there can be no confusion.

is denoted $E_{\mathcal{D}}$ so the graph is $G_{\mathcal{D}} = (V_{\mathcal{D}}, E_{\mathcal{D}})$. As established in [Puppe and Slinko, 2019], for any Condorcet domain $\mathcal{D}$ the graph $G_{\mathcal{D}}$ is a median graph [Mulder, 1978] and any median graph can be obtained in this way.

A domain $\mathcal{D}$ is called *connected* if its graph $G_{\mathcal{D}}$ is a connected subgraph of the permutahedron [Puppe and Slinko, 2019]; we note that domains $CD_{3,t}$ and $CD_{3,b}$ are connected but $CD_{3,m}$ is not. Danilov et al. [2012] called a domain of maximal width *semi-connected* if the two completely reversed orders can be connected by a path of vertices that is also a path in the permutahedron. They proved that a maximal Condorcet domain of maximal width is semi-connected if and only if it is a peak-pit domain. Puppe [2017] showed that for a maximal Condorcet domain semi-connectedness implies direct connectedness (Proposition A2) which means that any two linear orders in the domain are connected by a shortest possible (geodesic) path.

Finally, we give two more definitions that express two properties of Condorcet domains.

Definition 5. *We call a Condorcet domain $\mathcal{D}$* ample *if for any pair of alternatives $a, b \in A$ the restriction $\mathcal{D}_{\{a,b\}}$ of this domain to $\{a, b\}$ has two distinct orders, that is, $\mathcal{D}_{\{a,b\}} = \{ab, ba\}$.*

Definition 6 ([Slinko, 2019]). *A Condorcet domain $\mathcal{D}$ is called* copious *if for any triple of alternatives $a, b, c \in A$ the restriction $\mathcal{D}_{\{a,b,c\}}$ of this domain to this triple has four distinct orders, that is, $|\mathcal{D}_{\{a,b,c\}}| = 4$.*

Of course, any copious Condorcet domain is ample. We note that, if a domain $\mathcal{D}$ is copious, then it satisfies a unique set of never conditions (1).

Definition 7. *A complete set of never-conditions* (1) *is said to satisfy the* alternating scheme*, if for all $1 \le i < j < k \le n$ either*

(1) *$jN_{\{i,j,k\}}3$, if j is even, and $jN_{\{i,j,k\}}1$, if j is odd, or*
(2) *$jN_{\{i,j,k\}}1$, if j is even, and $jN_{\{i,j,k\}}3$, if j is odd.*

Following Galambos and Reiner [2008] we denote these domains as F_n and $\overline{F_n}$ and call Fishburn's domains [Danilov et al., 2012]. The second domain is flip-isomorphic to the first so we consider only the first one.

In particular, $F_2 = \{12, 21\}$, $F_3 = \{123, 213, 231, 321\}$ and

$$F_4 = \{1234, 1243, 2134, 2143, 2413, 2431, 4213, 4231, 4321\}.$$

[Galambos and Reiner, 2008] give the exact formula for the cardinality of F_n:

$$|F_n| = (n+3)2^{n-3} - \begin{cases} (n - \frac{3}{2})\binom{n-2}{\frac{n}{2}-1} & \text{for even } n \\ (\frac{n-1}{2})\binom{n-1}{\frac{n-1}{2}} & \text{for odd } n \end{cases} \tag{2}$$

3 MAIN RESULTS

Let us start with an observation.

Proposition 2. *Let $\mathcal{D}$ be a semi-connected Condorcet domain of maximal width on the set of alternatives A. Then:*

(i) *For any $a \in A$ its restriction $\mathcal{D}'$ on $A' = A - \{a\}$ is also a semi-connected domain of maximal width.*
(ii) *$\mathcal{D}$ is copious peak-pit domain.*

Proof. (i) If w and $\bar{w}$ are two completely reversed linear orders in $\mathcal{D}$, then after removal of a, their images will still be completely reversed.

Let u, v be two vertices in $G_{\mathcal{D}}$ which are neighbouring vertices in the permutahedron. Then v differs from u by a swap of neighbouring alternatives. Let u', v' be their images under the natural mapping of $\mathcal{D}$ onto $\mathcal{D}'$. If one of these swapped alternatives was a, then $u' = v'$. If not, u', v' will still differ by a swap of neighbouring alternatives. Hence $\mathcal{D}'$ is semi-connected.

(ii) Let $a, b, c \in A$ and let $\mathcal{D}''$ be restriction of $\mathcal{D}$ onto $\{a, b, c\}$. Since $\mathcal{D}$ is of maximal width, the same can be said about $\mathcal{D}''$ and without loss of generality we may assume that $\mathcal{D}''$ contains abc and cba. By (i) $\mathcal{D}''$ is semi-connected and hence there will be two intermediate orders in $\mathcal{D}''$ connecting abc and cba. These would be either acb and cab or bac and bca. Thus, $\mathcal{D}''$ has four linear orders, and, hence, $\mathcal{D}$ is copious domain satisfying $bN_{\{a,b,c\}}1$ and $bN_{\{a,b,c\}}3$, respectively. Hence it is a peak-pit domain. □

3.1 Danilov-Karzanov-Koshevoy construction and its generalisation

Let us now start describing the Danilov-Karzanov-Koshevoy construction [Danilov et al., 2012]. In fact, this will be a generalisation of their construction since in our construction two arbitrary linear orders are involved.

Definition 8. *We will call a linear order $w = c_1 \dots c_{n+m}$ a* shuffle *of $u = a_1 \dots a_m$ and $v = b_1 \dots b_n$, if:*

1. *for every $k = 1, \dots, n$ the initial segment $c_1 \dots c_k$ of w is written for some particular $s \le k$ with all of the $\{a_1, \dots, a_s\}$ and all of the $\{b_1, \dots, b_{k-s}\}$ (in particular, $c_1 = a_1$ or $c_1 = b_1$);*
2. *If the initial segment $c_1 \dots c_k$ of w is written with $\{a_1, \dots, a_s\}$ and $\{b_1, \dots, b_{k-s}\}$ for some $s \le k$, then c_{k+1} is either a_{s+1} or b_{k-s+1}.*

Given two linear orders u and v, we define a domain $u \oplus v$ as the set of all shuffles of u and v. It is clear from definition that $u \oplus v = v \oplus u$. The cardinality of this domain is $|u \oplus v| = \binom{n+m}{m}$.

Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be two Condorcet domains on disjoint sets of alternatives A and B, respectively. We define a *concatenation* of these domains as the domain

$$\mathcal{D}_1 \odot \mathcal{D}_2 = \{xy \mid x \in \mathcal{D}_1 \text{ and } y \in \mathcal{D}_2\}$$

on $A \cup B$. It is immediately clear that $\mathcal{D}_1 \odot \mathcal{D}_2$ is also a Condorcet domain of cardinality $|\mathcal{D}_1 \odot \mathcal{D}_2| = |\mathcal{D}_1||\mathcal{D}_2|$. We have only to check that one of the never-conditions is satisfied for triples $\{a_1, a_2, b\}$ where $a_1, a_2 \in A$ and $b \in B$ (for triples $\{a, b_1, b_2\}$ the argument will be similar). The restriction $(\mathcal{D}_1 \odot \mathcal{D}_2)|_{\{a_1, a_2, b\}}$ will contain at most two linear orders a_1a_2b and a_2a_1b, which is consistent both with never-top and never-bottom conditions.

Theorem 1. *Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be two Condorcet domains on disjoint set of alternatives A and B. Let $u \in \mathcal{D}_1$ and $v \in \mathcal{D}_2$ be arbitrary linear orders. Then*

$$(\mathcal{D}_1 \otimes \mathcal{D}_2)(u, v) := (\mathcal{D}_1 \odot \mathcal{D}_2) \cup (u \oplus v)$$

is a Condorcet domain. Moreover, if $\mathcal{D}_1$ and $\mathcal{D}_2$ are peak-pit domains, so is $(\mathcal{D}_1 \otimes \mathcal{D}_2)(u, v)$.

Proof. Let us fix u and v in this construction and denote $(\mathcal{D}_1 \otimes \mathcal{D}_2)(u, v)$ as simply $\mathcal{D}_1 \otimes \mathcal{D}_2$. If $a, b, c \in A$, then $(\mathcal{D}_1 \otimes \mathcal{D}_2)_{\{a,b,c\}} = (\mathcal{D}_1)_{\{a,b,c\}}$, i.e., the restriction of $\mathcal{D}_1 \otimes \mathcal{D}_2$ onto $\{a, b, c\}$ is the same as the restriction of $\mathcal{D}_1$ onto $\{a, b, c\}$. Hence $\mathcal{D}_1 \otimes \mathcal{D}_2$ satisfies the same never condition for $\{a, b, c\}$ as $\mathcal{D}_1$. For $x, y, z \in B$ the same thing happens.

Suppose now $a, b \in A$ and $x \in B$. Then $(\mathcal{D}_1 \odot \mathcal{D}_2)_{\{a,b,x\}} \subseteq \{abx, bax\}$. Let also $u_{\{a,b\}} = \{ab\}$. Then $(u \oplus v)_{\{a,b,x\}} = \{abx, axb, xab\}$, hence

$$(\mathcal{D}_1 \otimes \mathcal{D}_2)_{\{a,b,x\}} \subseteq \{abx, bax, axb, xab\}, \tag{3}$$

thus $\mathcal{D}_1 \otimes \mathcal{D}_2$ satisfies $aN_{\{a,b,x\}}3$. For $a \in A$ and $x, y \in B$ we have $(\mathcal{D}_1 \odot \mathcal{D}_2)_{\{a,x,y\}} \subseteq \{axy, ayx\}$. Let also $v_{\{x,y\}} = \{xy\}$. Then $(u \oplus v)_{\{a,x,y\}} = \{axy, xay, xya\}$, hence

$$(\mathcal{D}_1 \otimes \mathcal{D}_2)_{\{a,x,y\}} \subseteq \{axy, ayx, xay, xya\}, \tag{4}$$

thus $\mathcal{D}_1 \otimes \mathcal{D}_2$ satisfies $yN_{\{a,x,y\}}1$. □

Note: The inequalities (3) and (4) become equalities if for any $i \in \{1,2\}$ and any $a, b \in \mathcal{D}_i$ we have $(\mathcal{D}_i)|_{\{a,b\}} = \{ab, ba\}$, i.e., if $\mathcal{D}_1$ and $\mathcal{D}_2$ are ample.

Proposition 3. *If $|A| = m$ and $|B| = n$, then for any $u \in \mathcal{D}_1$ and $v \in \mathcal{D}_2$*

$$|(\mathcal{D}_1 \otimes \mathcal{D}_2)(u,v)| = |\mathcal{D}_1||\mathcal{D}_2| + \binom{n+m}{m} - 1. \tag{5}$$

Proof. We have $|\mathcal{D}_1 \otimes \mathcal{D}_2| = |\mathcal{D}_1||\mathcal{D}_2|$ and $|u \oplus v| = \binom{n+m}{m}$. These two sets have only one linear order in common which is uv. This proves (5). □

Proposition 4. *Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be of maximal width with $u, \bar{u} \in \mathcal{D}_1$ and $v, \bar{v} \in \mathcal{D}_2$. Then $(\mathcal{D}_1 \otimes \mathcal{D}_2)(u,v)$ is also of maximal width. If $\mathcal{D}_1$ and $\mathcal{D}_2$ are semi-connected, then so is $(\mathcal{D}_1 \otimes \mathcal{D}_2)(u,v)$.*

Proof. Since $\mathcal{D}_1$ and $\mathcal{D}_2$ are of maximal width, we have $\bar{u} \in \mathcal{D}_1$ and $\bar{v} \in \mathcal{D}_2$. Hence $\bar{u}\bar{v} \in \mathcal{D}_1 \odot \mathcal{D}_2$. We also have $vu \in u \oplus v$, and $\overline{vu} = \bar{u}\bar{v}$, hence $(\mathcal{D}_1 \otimes \mathcal{D}_2)(u,v)$ has maximal width. To prove the last statement we note that $\bar{u}\bar{v}$ can be connected to uv (which belongs both to $\mathcal{D}_1 \otimes \mathcal{D}_2$ and to $u \oplus v$) by a geodesic path and uv in turn can be connected to vu by a geodesic path within $u \oplus v$. □

If both $\mathcal{D}_1$ and $\mathcal{D}_2$ have maximal width, it is not true, however, that $(\mathcal{D}_1 \otimes \mathcal{D}_2)(u,v)$ will have maximal width for any $u \in \mathcal{D}_1$ and $v \in \mathcal{D}_2$. Let us take, for example, $\mathcal{D}_1 = \{x = ab, \bar{x} = ba\}$ and $\mathcal{D}_2 = \{u = cde, v = dec, w = dce, \bar{u} = edc\}$. Then $(\mathcal{D}_1 \otimes \mathcal{D}_2)(x,u)$ has maximal width while $(\mathcal{D}_1 \otimes \mathcal{D}_2)(x,v)$ does not since $\bar{v} \notin \mathcal{D}_2$. In particular,

$$(\mathcal{D}_1 \otimes \mathcal{D}_2)(x,u) \ncong (\mathcal{D}_1 \otimes \mathcal{D}_2)(x,v).$$

This indicates that the construction of the tensor product may be useful in description of Condorcet domains which do not satisfy the requirement of maximal width.

Proposition 5. *Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be two Condorcet domains on disjoint set of alternatives A and B. Let $u \in \mathcal{D}_1$ and $v \in \mathcal{D}_2$ be arbitrary linear orders. Then*

(i) *$(\mathcal{D}_1 \otimes \mathcal{D}_2)(u,v)$ is connected,whenever $\mathcal{D}_1$ and $\mathcal{D}_2$ are;*
(ii) *$(\mathcal{D}_1 \otimes \mathcal{D}_2)(u,v)$ is copious, whenever $\mathcal{D}_1$ and $\mathcal{D}_2$ are.*

Proof. (i) Domain $u \oplus v$ is obviously connected and connected by a single swap of neighboring alternatives to uv which belongs to $\mathcal{D}_1 \odot \mathcal{D}_2$. The latter is also connected since $\mathcal{D}_1$ and $\mathcal{D}_2$ are.

(ii) follows from (3) and (4) since in this case, as was noted before, these inequalities become equalities. □

Proposition 6.

$$(F_2(a,b) \otimes F_2(c,d))(ab, cd) \cong F_4(b,a,d,c). \tag{6}$$

Proof. We list orders of this domain as columns of the following matrix

$$[F_2(a,b) \odot F_2(c,d) \mid ab \oplus cd] = \left[\begin{array}{cccc|ccccc} a & a & b & b & a & a & c & c & c \\ b & b & a & a & c & c & a & a & d \\ c & d & c & d & b & d & b & d & a \\ d & c & d & c & d & b & d & b & b \end{array}\right].$$

We see that the following never conditions are satisfied: $aN_{\{a,b,c\}}3, aN_{\{a,b,d\}}3, dN_{\{a,c,d\}}1, dN_{\{b,c,d\}}1$. Hence the mapping $1 \to b$, $2 \to a$, $3 \to d$ and $4 \to c$ is an isomorphism of F_4 onto the tensor product. □

The isomorphism (6) is very nice but unfortunately for larger m, n we have $F_m \otimes F_n \not\cong F_{m+n}$. Moreover, it appears that for two maximal Condorcet domains $\mathcal{D}_1$ and $\mathcal{D}_2$ on sets A and B, respectively, $\mathcal{D}_1 \otimes \mathcal{D}_2$ may be not maximal on $A \cup B$. Here is an example.

Example 2. *Let us calculate* $\mathcal{E} := F_3(1,2,3) \otimes F_2(4,5)(321,54)$*:*

$$\left[\begin{array}{cccccccc|ccccccccc} 1 & 2 & 2 & 3 & 1 & 2 & 2 & 3 & 3 & 3 & 5 & 3 & 3 & 3 & 5 & 5 & 5 \\ 2 & 1 & 3 & 2 & 2 & 1 & 3 & 2 & 2 & 5 & 3 & 2 & 5 & 5 & 3 & 3 & 4 \\ 3 & 3 & 1 & 1 & 3 & 3 & 1 & 1 & 5 & 2 & 2 & 5 & 2 & 4 & 2 & 4 & 3 \\ 4 & 4 & 4 & 4 & 5 & 5 & 5 & 5 & 1 & 1 & 1 & 4 & 4 & 2 & 4 & 2 & 2 \\ 5 & 5 & 5 & 5 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 1 & 1 & 1 & 1 & 1 & 1 \end{array}\right].$$

There are 17 linear orders in this domain. It is known, however, that F_5 *has 20 [Fishburn, 1996] but this fact alone does not mean non-maximality of* $\mathcal{E}$*. By Proposition 5 this domain is copious. By its construction it satisfies just three inversion triples:*

$$[1,2,4], \quad [1,3,4], \quad [2,3,4].$$

Now we see that there are two more linear orders 23514 *and* 23541 *that satisfy these conditions.*

3.2 On Fishburn's hypothesis

We will further write $(F_k \otimes F_m)(u,v)$ simply as $F_k \otimes F_m$, when $u \in F_k$ and $v \in F_m$ are chosen so that $(F_k \otimes F_m)(u,v)$ has maximal width. We note that equation (6) is just a one of a kind since $F_2 \otimes F_3 \not\cong F_5$ already.

Our calculations, using formulas (2) and (5) show that

$$|F_n \otimes F_n| < |F_{2n}|$$

for $2 < n \le 19$ but $4611858343415 = |F_{20} \otimes F_{20}| > |F_{40}| = 4549082342996$. Earlier, [Danilov et al., 2012] showed that $|F_{21} \otimes F_{21}| > F_{42}$ disproving an old Fishburn's hypothesis that F_n is the largest peak-pit Condorcet domain on n alternatives [Fishburn, 1996, Galambos and Reiner, 2008].

4 CONCLUSION AND FURTHER RESEARCH

Operations over Condorcet domains are useful in many respects. The Danilov-Karzanov-Koshevoy construction is especially useful since it converts smaller peak-pit Condorcet domains into larger peak-pit domains. Fishburn's replacement scheme [Fishburn, 1996] also produces larger domains but without preserving peak-pittedness.

Now that we know that $g(n) > |F_n|$, the question whether or not $f(n) = g(n)$ comes to the fore. Fishburn's replacement scheme maybe instrumental in obtaining the answer (if it is negative).

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